A Finitely Additive Version of a Convergence Theorem of Lévy

by

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Technical Report No. 222

November 1973

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*Research partially supported by Army Grant DA-ARO-D-31-124-70-G-102.

In the conventional theory of probability, we have the following convergence theorem for the sums of mutually independent random variables, i.e., suppose that $X_1, X_2 \ldots$ are mutually independent random variables defined on a probability space $(\Omega, \mathfrak{F}, P)$ then the following two statements are equivalent.

(i) $\sum_{j=1}^{n} X_j$ converges to S almost surely as $n \to \infty$

(ii) $\sum_{j=1}^{n} X_j$ converges to S in probability as $n \to \infty$

where S is a random variable defined on the probability space $(\Omega, \mathfrak{F}, P)$. In this paper, we will state and prove a counterpart (Theorem 2-3) of this theorem without the countable additivity assumption.

1. Basic definitions and some useful lemmas.

In this section, we will give some preliminary definitions and then state some useful lemmas. Lemma 1-1, 1-2 and more details are available in [3]. Throughout this paper, we will let F be a non-empty set with the discrete topology and $H = F^{\infty} = F \times F \times F \times \ldots$, with the product topology.

Definition 1-1.

Suppose that, for each $n = 1, 2, 3 \ldots,$ γ_n is a finitely additive probability measure defined on the class of all subsets of F. Let σ be the strategy (cf. [2] pp. 11-12) defined by $\sigma_0 = \gamma_1$, and, for $n = 1, 2, \ldots,$ and $f_1, \ldots, f_n$ elements of F. $\sigma_n(f_1, \ldots f_n) = \gamma_{n+1}$. Then σ is called an independent strategy on H and sometimes written $\sigma = \gamma_1 \times \gamma_2 \times \ldots$.

Definition 1-2.

A sequence $\{Y_n | n=1, 2,\ldots\}$ of real-valued functions defined on H is called a sequence of coordinate mappings defined on H, if, for each $n = 1, 2,\ldots,$ the function Y_n depends only on the n^{th} coordinate.

It was shown that (for general setting, see [3]), if F, H and σ are defined as above, then there exists a field $a(\sigma)$ of subsets of H such that $a(\sigma)$ contains all Borel subsets of H and σ is a finitely additive probability measure defined on $a(\sigma)$ with some nice properties. Therefore, we can consider $(H, a(\sigma), \sigma)$ as a finitely additive probability space.

Definition 1-3.

Let $Y_1, Y_2,\ldots, Y$ be real-valued functions defined on H. We say that Y_n converges to Y almost surely as $n \to \infty$, if the set $K = [h| \lim_{n\to\infty} Y_n(h) = Y(h)]$ is in $a(\sigma)$ and $\sigma(K) = 1$. We say that Y_n converges to Y in σ-probability as $n \to \infty$ if, for each $\epsilon > 0$, each $n = 1, 2, 3,\ldots,$ there exists a set $L_n(\epsilon)$ in $a(\sigma)$ such that the set $K_n(\epsilon) = [h| \, |Y_n(h) - Y(h)| > \epsilon]$ is a subset of $L_n(\epsilon)$ and $\lim_{n\to\infty} \sigma(L_n(\epsilon)) = 0$. We say that $\{Y_n | n=1, 2,\ldots\}$ is a fundamental sequence in σ-probability if, for each $\epsilon > 0$, each $n = 1, 2, 3\ldots$ and each $m = 1, 2,\ldots,$ there exists a set $L_{n,m}(\epsilon)$ in $a(\sigma)$ such that the set $K_{n,m}(\epsilon) = [h| \, |Y_n(h) - Y_m(h)| > \epsilon]$ is a subset of $L_{n,m}(\epsilon)$ and $\lim_{\substack{n\to\infty\\ m\to\infty}} \sigma(L_{n,m}(\epsilon)) = 0$.

Let $N_1, N_2,\ldots$ be positive integers, and for each $j = 1, 2,\ldots,$ let C_j be a subset of F^{N_j} (N_j factors). Set $r_1 = 1$, $r_n = \sum_{j=1}^{n-1} N_j + 1$ for $n = 2, 3, 4,\ldots$. Set $t_n = \sum_{j=1}^{n} N_j$ for $n = 1, 2, 3,\ldots$. For each

$n = 1, 2, \ldots,$ let K_n be a subset of H defined by $K_n = [h | h=(f_1, \ldots, f_{r_n}, \ldots, f_{t_n}, \ldots), (f_{r_n}, \ldots, f_{t_n}) \in C_n]$ i.e., $K_n = F^{r_n - 1} \times C_n \times H$, $n = 1, 2, \ldots$.

Lemma 1-1.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ is an independent strategy on H and $\{K_1, K_2, \ldots\}$ is a sequence of subsets of H defined above. Then $\sigma(\bigcap_{j=1}^{\infty} K_j) = \prod_{j=1}^{\infty} \sigma(K_j)$.

Proof: See page 35 of [3].

Lemma 1-2.

Suppose that σ and $\{K_n | n=1, 2, 3, \ldots\}$ are defined as in Lemma 1-1, then

(i) $\sum_{j=1}^{\infty} \sigma(K_j) < \infty$ if and only if $\sigma([K_n \text{ i,o}(n)]) = 0$

(ii) $\sum_{j=1}^{\infty} \sigma(K_j) = \infty$ if and only if $\sigma([K_n \text{ i,o}(n)]) = 1$

where $[K_n \text{ i,o}(n)] = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} K_n = [h | h \in K_n$ for infinitely many $n]$.

Proof: See page 39 of [3].

Lemma 1-3.

Suppose that $\sigma = \gamma_1 \times \gamma_2 \times ,,,$ is an independent strategy on H and $\{Y_n | n=1, 2, 3 \ldots\}$ is a sequence of coordinate mappings defined on H. Let $S_0 = 0$, and, for each $n = 1, 2, 3 \ldots$ $S_n = \sum_{j=1}^{n} Y_j$. Then, if $\epsilon > 0$, $\delta > 0$, M, N are two integers such that $0 \leq M < N < \infty$, and

$\max_{M<n\leq N} \sigma([h| \; |S_N(h) - S_n(h)| > \epsilon]) < \delta$,

$$\sigma([h| \max_{M<n\leq N} |S_n(h) - S_M(h)|> 2\epsilon])$$

$$\leq \frac{1}{1-\delta}\, \sigma([h|\ |S_N(h) - S_M(h)| > \epsilon]).$$

Proof: This result does not seem to have been stated before. The proof is essentially the same as the one in the conventional theory of probability (see page 45 of [1]) and we omit it.

2. Main Theorems.

In the countably additive theory of probability, the almost sure convergence implies the convergence in probability but it is not true in the finitely additive theory of probability. (see Example 1 below). This section is devoted to proving the equivalence of these two convergences for the sums of coordinate mappings with respect to independent strategies.

Theorems 2-1, 2-2, and 2-3 are new. Theorem 2-2 seems to be the central result and Theorem 2-3 is the counterpart of the Lévy Convergence Theorem with respect to an independent strategy and coordinate mappings. In this section, we will assume that σ is an independent strategy on H, $\{Y_n|n=1, 2...\}$ is a sequence of coordinate mappings defined on H, and $S_0 = 0$, $S_n = \sum_{j=1}^{n} Y_j$ for each $n = 1,2...$.

Theorem 2-1.

Suppose that σ, $\{Y_n|n = 1,2...\}$, and $\{S_n|n=0, 1,2...\}$ are defined as above. Then the following two statements are equivalent

(i) $\{S_n | n = 1, 2\ldots\}$ is a fundamental sequence in σ-probability

(ii) $\sigma([h| \lim_{n \to \infty} S_n(h)$ exists and is finite$]) = 1$

Proof: (i) $\Rightarrow$ (ii)

For each $j = 1, 2, \ldots$ let $\epsilon_j = \frac{1}{(1+j)^{1+\alpha}}$ where α is a positive constant. By the definition of $\{S_n|n=1,2\ldots\}$ being fundamental in σ-probability we can choose a strictly increasing sequence $\{N_j | j = 1, 2\ldots\}$ of positive integers such that, for each $j = 1,2,\ldots$ $\sigma([h| \, |S_n(h) - S_m(h)| > \epsilon_j]) \le \epsilon_j$ if $m, n \ge N_j$. Let $D_j = \max_{N_j < n \le N_{j+1}} |S_n - S_{N_j}|$, $L_j = [h|D_j(h) > 2\epsilon_j]$, then D_j and L_j depend only on the $N_j + 1^{st} \ldots N_{j+1}^{th}$ coordinates, $j = 1, 2, 3\ldots$. By Lemma 1-3,

$$\sigma(L_j) \le \frac{1}{1-\epsilon_j} \sigma([h| \, | S_{N_{j+1}}(h) - S_{N_j}(h)| > \epsilon_j])$$

$$\le \frac{\epsilon_j}{1-\epsilon_j} \le 2\epsilon_j \quad (\text{since } \epsilon_j \le \tfrac{1}{2} \ \forall\ j = 1, 2,\ldots).$$

Since $\sum_{j=1}^{\infty} \epsilon_j = \sum_{j=1}^{\infty} \frac{1}{(j+1)^{1+\alpha}} < \infty$,

$\sum_{j=1}^{\infty} \sigma(L_j) \le \sum_{j=1}^{\infty} 2\epsilon_j < \infty$. By Lemma 1-2, $\sigma([L_n \ i,0(n)]) = 0$.

Notice that the set $[h| \lim_{n \to \infty} S_n(h)$ exists and is finite$]$ contains the set $[L_n \ i,0(n)]^c$.

So $\sigma([h| \lim_{n \to \infty} S_n(h)$ exists and is finite$]) = 1$.

(ii) $\Rightarrow$ (i).

Suppose that $\{S_n | n=1,2\ldots\}$ is not a fundamental sequence in σ-probability, then there exists a strictly increasing sequence $\{N_j | j=1, 2\ldots\}$ of positive integers and two positive real numbers ϵ, δ such that $\sigma([h| \ |S_{N_{j+1}}(h) - S_{N_j}(h)| > \epsilon\ ]) \geq \delta$ for all $j = 1, 2, 3\ldots$. Notice that the set $[h| \ |S_{N_{j+1}}(h) - S_{N_j}(h)| > \epsilon]$ depends only on the $N_j+1^{st}, \ldots, N_{j+1}{}^{th}$ coordinates $(j = 1, 2, \ldots)$ hence, by Lemma 1-2, we should have $\sigma([h| \ |S_{N_{j+1}}(h) - S_{N_j}(h)| > \epsilon\ i,o(j)]) = 1$. But the set $[h|\lim_{n\to\infty} S_n(h)$ exists and is finite] and the set $[h| \ |S_{N_{j+1}}(h) - S_{N_j}(h)| > \epsilon\ i,o\ (j)]$ are disjoint, and $\sigma([h|\lim_{n\to\infty} S_n(h)$ exists and is finite $]) = 1$. So we get a contradiction.

Therefore $\{S_n | n = 1, 2, 3\ldots\}$ must be a fundamental sequence in σ-probability.

Theorem 2-2.

Suppose that σ, $\{Y_n | n = 1, 2, \ldots\}$, $\{S_n | n = 0, 1,2\ldots\}$ are defined as above, and S is a Borel measurable function defined on H. Then we have the following results.

(i) If S_n converges to S in σ-probability as $n \to \infty$, then $\{S_n | n=1, 2, \ldots\}$ is a fundamental sequence in σ-probability.

(ii) If $\{S_n | n=1, 2, 3\ldots\}$ is a fundamental sequence in σ-probability, then, for any $\epsilon > 0$, $\delta > 0$, there exists a positive integer N such that

$$\sigma([h|\sup_{n<K<\infty}|S_K(h)-S_n(h)|\geq\epsilon])\leq\delta \quad \text{whenever} \quad n\geq N.$$

Proof: "(i) ⇒ (ii)" is obvious and we omit it.

(ii) ⇒ (i)

For each $j=1,2,\ldots$, let $\epsilon_j=\frac{\epsilon}{2^{j+1}}$ and δ_j be a positive real number such that $\prod_{j=1}^{\infty}(1-\frac{\delta_j}{1-\delta_j})\geq 1-\delta$. Since $\{S_n|n=1,2,\ldots\}$ is a fundamental sequence in σ-probability, there exists, for each $j=1,2,\ldots$, an $N_j\geq 1$ such that

$$\sigma([h|\,|S_n(h)-S_m(h)|\geq\epsilon_j])\leq\delta_j$$

if $n\geq N_j$, $m\geq N_j$, we can and do assume that $1\leq N_1<N_2<\ldots$.

By Lemma 1-3, we have, for each $j=1,2,\ldots$,

$$\sigma([h|\max_{N_j<K\leq N_{j+1}}|S_K(h)-S_{N_j}(h)|\geq 2\epsilon_j])$$

$$\leq\frac{1}{1-\delta_j}\sigma([h|\,|S_{N_{j+1}}(h)-S_{N_j}(h)|\geq\epsilon_j])$$

$$\leq\frac{\delta_j}{1-\delta_j}.$$

Now, for each $j=1,2,3\ldots$, let

$$L_j=[h|\max_{N_j<m\leq N_{j+1}}|S_m(h)-S_{N_j}(h)|<2\epsilon_j].$$

Then $\sigma(L_j)\geq 1-\frac{\delta_j}{1-\delta_j}$.

Notice that the set L_j depends only on the $N_j+1^{st},\ldots,N_{j+1}{}^{th}$ coordinates.

$(j = 1, 2, 3 \ldots)$ Hence, by Lemma 1-1, $\sigma(\bigcap_{j=1}^{\infty} L_j) \geq \prod_{j=1}^{\infty}(1 - \frac{\delta_j}{1-\delta_j}) \geq 1 - \delta$.

Now, set $N = N_1$, then $[h| \sup_{n<m<\infty} |S_m(h) - S_n(h)| < \epsilon] \supseteq \bigcap_{j=1}^{\infty} L_j$ if $n \geq N$.

Therefore

$$\sigma([h| \sup_{n<m<\infty} |S_m(h) - S_n(h)| < \epsilon]) \geq 1 - \delta \quad \text{if} \quad n \geq N,$$

which is equivalent to the statement of the theorem.

<u>Theorem 2-3</u>.

Suppose that σ, $\{Y_n | n = 1, 2, \ldots\}$, $\{S_n | n = 0, 1, 2 \ldots\}$ are defined as above and S is a Borel measurable function defined on H. Then, we have

(i) If S_n converges to S almost surely as $n \to \infty$, then S_n converges to S in σ-probability as $n \to \infty$.

(ii) If S_n converges to S in σ-probability as $n \to \infty$, then $\sigma([h| \lim_{n\to\infty} S_n(h)$ esists and is finite$]) = 1$ and $\sigma([h| \; |\lim_{n\to\infty} S_n(h) - S(h)| \geq \epsilon]) = 0$ for all $\epsilon > 0$.

Proof: The statement (i) is implied by Theorem 2-1, the (ii) of Theorem 2-2, and the fact

$$\sigma([h| \lim_{n \to \infty} S_n(h) = S(h)]) = 1 .$$

The first part of the statement (ii) is implied by the (i) of Theorem 2-2 and Theorem 2-1. Now, we prove the second part of the statement (ii).

Let $S^*(h) = \overline{\lim}_{n\to\infty} S_n(h) \quad \forall\, h \in H$, S^* is Borel measurable and

$$\sigma([h| -\infty < S^*(h) < \infty, \lim_{n\to\infty} S_n(h) = S^*(h)]) = 1$$

By the part (i) of the theorem, we have that S_n converges to S^* in σ-probability as $n \to \infty$. Now, for any $\epsilon > 0$,

$$[h| \; |S^*(h) - S(h)| > \epsilon]$$

$$\subseteq [h| \; | S^*(h) - S_n(h)| > \tfrac{\epsilon}{2}]$$

$$\cup [h| \; | S^*(h) - S_n(h)| > \tfrac{\epsilon}{2}]$$

for each $n = 1, 2, \ldots$. Hence

$$\sigma([h| \; |S^*(h) - S(h)| > \epsilon \;])$$

$$\leq \lim_{n\to\infty} \{\sigma([h| \; | S^*(h) - S_n(h)| > \tfrac{\epsilon}{2}])$$

$$+ \sigma([h| \; |S(h) - S_n(h))] > \tfrac{\epsilon}{2}])\} = 0$$

Corollary 2-1.

Suppose that $\sigma_1, \{Y_n | n = 1, 2 \ldots\}$, $\{S_n | \; n = 0, 1, 2 \ldots\}$ are defined as above and $\{S_n | n = 1, 2, \ldots\}$ is a fundamental sequence in σ-probability. Then there exists a real-valued function S defined on H such that S_n converges to S almost surely as $n \to \infty$ and also S_n converges to S in σ-probability as $n \to \infty$.

Proof: By Theorem 2-1, we have $\sigma(A) = 1$ where $A = [h|\lim_{n\to\infty} S_n(h)$ exists and is finite]. Now, define the real-valued function S on H by

$$S(h) = \lim_{n\to\infty} S_n(h) \qquad \text{if } h \in A$$
$$= 0 \qquad \text{if } h \notin A .$$

Then it is obvious that S_n converges to S almost surely as $n \to \infty$. By Theorem 2-3, we have S_n converges to S in probability as $n \to \infty$.

3. Remarks.

1. In the statement (ii) of Theorem 2-3, we cannot let $\epsilon = 0$, i.e., it is not true that $\sigma([h|\lim_{n\to\infty} S_n(h) = S(h)]) = 1$. (See the example 2 below.)
2. In the conventional theory of probability, the following statement is true "suppose that $X_1, X_2, \ldots, X, Y$ are random variables defined on a probability space $(\Omega, \mathfrak{F}, P)$. Then, if X_n converges to X in probability as $n \to \infty$, and X_n converges to Y in probability as $n \to \infty$ too, $P(\{W|X(W) \neq Y(W)\}) = 0$". But this statement is false for our setting, the following is a counter example.

Example 1.

Suppose that $F = \{1, 2, 3\ldots\}$, $H = F^\infty = F \times F \times \ldots$ and γ is a finitely additive probability measure defined on the class of all subsets of F such that $\gamma(A) = 0$ if A is a finite subset of F. Let $\sigma = \gamma \times \gamma \times \gamma \times \ldots$. be an independent strategy on H, Y_1 be the real-values function defined on H by $Y_1(h) = f_1$ if $h = (f_1, f_2, \ldots) \in H$

and, for each $n = 2, 3, \ldots,$ Y_n be the real-valued function defined on H by $Y_n(h) = 1$ for all $h \in H$. Now, for each $n = 1, 2, \ldots$

$$Z_n = \frac{1}{n} \sum_{j=1}^{n} Y_j ,$$

then

$$\sigma([h|\ \lim_{n \to \infty} Z_n(h) = 1]) = 1$$

but

$$\sigma([h|\ |Z_n(h) - 1| > \epsilon]) = 1 \quad \text{for all} \quad n = 1, 2, \ldots \quad \text{and all} \quad \varepsilon > 0.$$

Hence Z_n does not converge to 1 in σ-probability as $n \to \infty$.

Example 2:

Suppose that $F = \{1, 2, 3\ldots\}$, $H = F^{\infty} = F \times F \times \ldots$ and γ is a finitely additive probability measure defined on the class of all subsets of F such that $\gamma(A) = 0$ if A is a finite subset of F. Let $\sigma = \gamma \times \gamma \times \ldots$ be an independent strategy on H, and for each $n = 1, 2, 3, \ldots$ Y_n be the real-valued function defined on H such that $Y_n(h) = \frac{1}{f_n}$ if $h = (f_1, f_2, \ldots) \in H$. Then, it is easy to check that $\sum_{j=1}^{n} Y_j$ converges to 0 in σ-probability as $n \to \infty$ and also converges to S in σ-probability as $n \to \infty$, where S is a real-valued function defined on H by

$$\begin{aligned} S(h) &= \sum_{j=1}^{\infty} Y_j(h) && \text{if } \sum_{j=1}^{\infty} Y_j(h) < \infty \\ &= 0 && \text{if } \sum_{j=1}^{\infty} Y_j(h) = \infty . \end{aligned}$$

But $\sigma([h|\ S(h) > 0]) = 1$ (" $\sum_{j=1}^{n} Y_j$ converges to S in σ-probability as $n \to \infty$" is implied by Theorem 2-3).

Acknowledgements

I would sincerely like to thank my advisor, Professor William D. Sudderth for his invaluable guidance and encouragement to me while working on this paper. I also want to thank Professor John R. Baxter for his helpful conversations about measure theory.

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